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OPTIMAL CONTROL OF AN UNMANNED AERIAL VEHICLE

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The main objective of this work is to develop algorithms and software solutions for finding optimal control strategies for an unmanned aerial vehicle, the mathematical model of which is represented by a system of ordinary differential equations. Using the maximum principle, an algorithm was developed to determine the optimal control. The presented methods and models are of high practical importance for managing various economic and technical systems.

Keywords: UAV, dynamics, mathematical model, controllability, stability.

ҰШҚЫШСЫЗ ҰШУ АППАРАТЫН ОҢТАЙЛЫ БАСҚАРУ

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Бұл жұмыстың негізгі міндеті-математикалық моделі қарапайым дифференциалдық теңдеулер жүйесімен ұсынылған дронды басқарудың оңтайлы стратегияларын іздеу үшін алгоритмдер мен бағдарламалық шешімдерді құру. Максимум принципін қолдана отырып, оңтайлы басқаруды анықтау үшін алгоритм жасалды. Ұсынылған әдістер мен модельдер әртүрлі экономикалық және техникалық жүйелерді басқарудың жоғары практикалық маңыздылығына ие.

Түйін сөздер: ҰҰА, динамика, математикалық модель, басқарылғыштық, тұрақтылық.

ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ БЕСПИЛОТНЫМ ЛЕТАТЕЛЬНЫМ АППАРАТОМ

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Главная задача данной работы заключается в создании алгоритмов и программных решений для поиска оптимальных стратегий управления беспилотным летательным аппаратом, чья математическая модель представлена системой обыкновенных дифференциальных уравнений. Используя принцип максимума, был разработан алгоритм для определения оптимального управления. Представленные методы и модели обладают высокой практической значимостью для управления различными экономическими и техническими системами.

Ключевые слова: БПЛА, динамика, математическая модель, управляемость, устойчивость.

Introduction. Unmanned aviation is actively developing worldwide due to the growing demand for lightweight, relatively inexpensive aircraft with high maneuverability, capable of performing a wide range of tasks. Unmanned aerial vehicles (UAVs) are successfully used in both military operations and civilian applications, including linearization [1-5].

The study of many aviation systems is reduced to the creation of mathematical models described by nonlinear ordinary differential equations. However, no universal solution methods have been developed for nonlinear systems. It is important to consider the nature of nonlinearities when studying such mathematical models.

Optimization of control is one of the key tasks in the theory of controlled dynamic systems. To design and operate aviation systems, it is necessary to achieve the goal with maximum efficiency, which requires minimizing a certain quality functional.

Materials and Methods. The work is devoted to the study of optimal control of the mathematical model of UAV dynamics:

$$\begin{cases} \dot{V} = g(n_{xa} - sin\Theta) \\ \dot{\Theta} = g(n_{ya}cos\gamma - cos\Theta)/V \\ \dot{\Psi} = -gn_{ya}sin\gamma/(Vcos\Theta) \\ \dot{x} = Vcos\Theta\cos\Psi \\ \dot{y} = Vsin\Theta \\ \dot{z} = -Vcos\Theta\sin\Psi \end{cases} \quad (1)$$

$$n_{xa} = \frac{Pcos\alpha - X_a}{mg},\ n_{ya} = \frac{Psin\alpha + Y_a}{mg} \quad (2)$$

Where:

- *x, y, z* are the coordinates of the aircraft's center of mass in the normal Earth coordinate system;
- V is the flight speed;
- Θ is the trajectory angle;
- Ψ is the course angle;
- α is the angle of attack;
- γ is the roll angle;
- P is the engine thrust;
- X_a is aerodynamic drag;
- Y_a is aerodynamic lift;
- m is the mass of the aircraft;
- g is gravitational acceleration;
- n_{xa} is the longitudinal overload;
- n_{ya} is the lateral overload (in flow axes of coordinates) [6-7].

Overloads n_{xa}, n_{ya} and the roll angle γ are taken as control variables in (1).

We introduce the following notation:

$$q = \begin{bmatrix} V \\ \Theta \\ \Psi \\ x \\ y \\ z \end{bmatrix},\ q_0 = \begin{bmatrix} V_0 \\ \Theta_0 \\ \Psi_0 \\ x_0 \\ y_0 \\ z_0 \end{bmatrix},\ q_1 = \begin{bmatrix} V_1 \\ \Theta_1 \\ \Psi_1 \\ x_1 \\ y_1 \\ z_1 \end{bmatrix} \tag{3}$$

The system of equations (1) is rewritten as the following system of nonlinear ordinary differential equations:

$$\dot{x} = f\left(x, t\right) + Bu(t) \tag{4}$$

Where:

- *f(x,t)* is an n-vector whose elements are continuously differentiable functions of their arguments;
- x is an n-dimensional system state vector;
- u is scalar control.

Control is subject to constraints:

$$u(t) \in U = \{u(t) :\ u(t) \in C[[t_0, t_1];-L \le u(t) \le L,\ t \in [t_0, t_1]\} \tag{5}$$

The problem is to find a control satisfying constraint (5) that transfers the system from the initial state

$$x\left(t_0\right) = x_0 \tag{6}$$

to the final specified state

$$x(t_1) = x_1 \tag{7}$$

within a fixed time $t_1 - t_0$.

For the quality assessment of system performance, the following criteria can be selected:

$$J = \int_{t_0}^{T} \left[u^*(t) R_0 u(t)\right] dt \tag{8}$$

In functional (8), R_0 is a positively definite *m*x*m* matrix. The required state at the final time can be specified as fixed (7) or changing (satisfying certain conditions).

$$\sum_{j=1}^{n} c_{ij} x_j(T) \le d_i, i = \overline{1, k} \tag{9}$$

The optimal control problem is considered, including control constraints (5) with fixed (7) or variable boundaries (8). Currently, solving such problems is accompanied by many mathematical difficulties.

Various formulations of optimal control problems will be considered. The task is to minimize the functional (8) under the constraints (4), (5), (6), and (7). The time moment T is considered given (fixed).

To solve the optimal control problem, we will construct the Hamilton function:

$$H\left(x(t), u, \psi(t), \psi_0\right) = u^*(t)R_0u(t) + (g(x,t) + Bu(t))^*\psi \qquad (10)$$

And form the conjugate system of differential equations:

$$\frac{d\psi}{dt} = -(\frac{\partial g(t)}{\partial t})^*(t)\psi(t),\ t \in [t_0, \quad T] \qquad (11)$$

The optimal control is determined by condition (5) and the maximum of the Hamiltonian:

$$u = \begin{cases} 0 & \text{если} \quad R_0^{-1}B\psi < 0 \\ R_0^{-1}B\psi & \text{если} \quad 0 \leq R_0^{-1}B\psi \leq u_{\max} \\ u_{\max} & \text{если} \quad R_0^{-1}B\psi > u_{\max} \end{cases} \qquad (12)$$

Theorem: Let the pair $(u(t), x(t))$, $t \in [t_0, \quad T]$ be the solution to the problem posed above. Then, there necessarily exists a vector function $\psi(t), t \in [t_0, \quad T]$ and a parameter ψ_0 such that:

1) $\psi_0 \leq 0$, $|\psi_0| + |\psi(t)| \neq 0,\ t \in [t_0, \quad T]$

2) The pair $x(t), \psi(t), \quad t \in [t_0, \quad T]$ is the solution to the boundary value problem for the system of differential equations (4) and the corresponding conjugate system of differential equations (11) with boundary conditions (6) and (7) and control (12).

Next, we study the problem of optimal control with fixed boundaries (6)-(7) and control constraints (5). Currently, solving such problems also encounters certain mathematical difficulties.

For the practical solution of the control optimization problem, penalty function methods and the gradient method are applied. To account for constraints at the end of the trajectory (7), we introduce a penalty function $_k = {}_k \sum_{i=1}^{n} \left[x(T) - x_T)\right]^2$, where $\{_k\}$ is a predefined positive sequence tending to infinity. We construct a new functional:

$$J_k = \int_{t_0}^{T} u^*(t)R_0u(t) + M_k \sum_{i=1}^{n} [x(T) - x_T]^2$$

The problem can be reformulated as follows: for a given value of the parameter *k*, find the optimal control minimizing the functional J_k subject to constraints (5)-(7). This problem belongs to the class of optimal control problems with a free right boundary and control constraints.

For it, we will construct the Hamiltonian function:

$$H_k = u^*(t)R_0u(t) + (g(x,t) + Bu(t))^*\psi_k$$

The following solution algorithm is proposed:

Step 1: Let $k = 0$.

Step 2: Calculate the optimal control for the k-th iteration using equation (12), where $_k$ is the solution to the conjugate system of differential equations (11), with the boundary condition at the end:

$$\Psi_k(T) = 2M_k \sum_{i=1}^{n} [x_k(T) - x_T] \tag{13}$$

and x_k is the solution of the original system (4) with initial conditions (6).

Step 3: Calculate the value of the functional J_k for the obtained x_k and u_k.

Step 4: If $|J_k - J_{k-1}| \leq \varepsilon$ proceed to step 5, otherwise, set $k = k + 1$ and go back to step 2. (Here, $\varepsilon > 0$ is the required accuracy of the calculation).

Step 5: The found pair (x_k, u_k) is the optimal solution.

To automate the process of finding the optimal control, a program "Optim_Upr.m" was written in MATLAB. As a result of its execution, we obtain the analytical form of the original and conjugate systems of differential equations, the form of the Hamiltonian function for the UAV mathematical model (1)-(3):

```
H=10.*ux^2+10.*uy^2+(9.80*ux-9.80*sin(O))*fi1+(5.15*uy-9.80*cos(O))/V*fi2
   -8.34*uy/V/cos(O)*fi3+V*cos(O)*cos(K)*fi4+V*sin(O)*fi5-1.*V*cos(O)*sin(
   K)*fi6

f1 = 9.80*ux - 9.80*sin(O);
f2 = (5.15*uy - 9.80*cos(O))/V;
f3 = -8.34*uy/V/cos(O);
f4 = V*cos(O)*cos(K);
f5 = V*sin(O);
f6 = -1.*V*cos(O)*sin(K);

fp1=(5.15*uy-9.80*cos(O))/V^2*fi2-8.34*uy/V^2/cos(O)*fi3-1.*cos(O)*cos(K)*
   fi4-1.*sin(O)*fi5+cos(O)*sin(K)*fi6

fp2=9.80*cos(O)*fi1-9.80*sin(O)/V*fi2+8.34*uy/V/cos(O)^2*fi3*sin(O)+V*sin(O)
   *cos(K)*fi4-1.*V*cos(O)*fi5-1.*V*sin(O)*sin(K)*fi6

fp3 = V*cos(O)*sin(K)*fi4 + V*cos(O)*cos(K)*fi6;

fp4 = 0;
fp5 = 0;
fp6 = 0;

H1 = -20.*ux - 9.80*fi1;
H2 = -20.*uy - 5.15/V*fi2 + 8.34/V/cos(O)*fi3;
```

Here f1,…,f6 is the view of the right side of the initial system of differential equations, fp1,…,fp6 is the view of the right side of the conjugate system.

The obtained results are transferred to a program written in Delphi, which performs numerical calculations to find the optimal control.

The results of the program are output to the file "rez.txt," a fragment of which is provided below:

```
T=1,00 nt=1000
-10,00<=U1<=10,00
-5,00<=U2<=5,00
x0 = 8,00;  = 1,00;  = 3,00;  = 5,00;  = 1,00;  = 3,00;
xk = 4,00;  = 0,50;  = 1,50;  = 2,50;  = 0,50;  = 1,50;
```

```
F1 ==   1009,53  11276,69  11276,69
F2 ==   1164,04  20148,99  10102,31
F3 ==   1191,48  40409,25  10874,49
F4 ==   1127,07  42853,26  5356,66
F5 ==   1250,00  184817,12  4551,07
F6 ==   1244,47  321606,47  3050,20
```

A total of 6 functionals F1,…,F6. were calculated. As can be seen from the last column, the difference between the specified and final points decreases.

Results and Discussion. Constructing the conjugate system of differential equations (11) for the original nonlinear system (1) is a rather labor-intensive process, especially when the dimensionality *n* increases, and is practically impossible for $n > 3$. Furthermore, when forming the Hamiltonian function, there is often a ”human factor” involved, which does not guarantee the correctness of the analytical calculations. Therefore, the automation of verifying the conditions of the theorems becomes relevant.

To address this issue, it is recommended to use computer algebra systems (CAS), such as MATLAB [8-9]. These systems provide a wide range of tools for working with algebraic expressions, from simple operations like calculation and differentiation to more complex ones like series expansion and integration. The application of such systems is relevant in various industries, including the aerospace industry.

Numerical calculations showed correspondence with experimental data. The results are also saved in text files, allowing the visualization of UAV dynamics in the form of one-dimensional graphs using MATLAB, for which a special program was developed. Based on the presented theory, an application was created [10].

Conclusions. The development of unmanned aviation requires the creation of optimal control methods for dynamic systems, which is crucial for enhancing the efficiency of unmanned aerial vehicle (UAV) management. This study confirms the necessity of finding optimal solutions in the presence of nonlinear equations and control constraints.

The article proposes a mathematical model of UAV dynamics described by a system of nonlinear differential equations. Several variants of the optimal control problem were solved with fixed and variable boundaries, as well as by using penalty functions and the gradient method to find optimal trajectories. The solution to the optimal control problem involves minimizing a functional under given constraints, which requires significant mathematical computations.

The application of a computer algebra system (MATLAB) enabled the automation of complex differential equation system calculations and minimized the human factor, which is important for increasing the accuracy of computations. Numerical calculations confirmed the correctness of the proposed theoretical model. The results showed consistency between calculated data and real experimental observations, indicating the applicability of the model for real UAV control systems.

The developed ”Optim_Upr.m” program in MATLAB, along with the application created for visualizing the results, simplifies solving UAV dynamics control tasks and provides results in the form of text files and graphs. This opens up opportunities for the further application of these methods in real-world conditions.

The continued use and development of the proposed algorithms and software can significantly improve the efficiency of UAV control systems, enabling them to adapt to changing real-time conditions and be applied in various industries, including the aviation and space sectors.

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